Eur. Phys. J. C 55, 337–342 (2008)
DOI 10.1140/epjc/s10052-008-0577-7

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Gauss–Codazzi formalism to brane-world within Brans–Dicke theory

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Received: 15 January 2008 / Revised version: 20 February 2008 /
Published online: 15 April 2008 –

Abstract. We apply the Gauss–Codazzi formalism to brane-worlds within the framework of Brans–Dicke gravity. The compactification is taken from six to five dimensions in order to formalize brane-world models with hybrid compactification in scalar tensor theories.

PACS. 04.50.+h; 98.80.Cq

1 Introduction

Advances in the establishment of string theory point to a multidimensional world [1]. This claim, from among other problems such as the hierarchy one, originated several works in which our universe is understood as a membrane (the brane), or a submanifold, embedded into a higher dimensional spacetime (the bulk) [2–9]. In the main compactification of M-theory there is a quite interesting symmetry of the orbifold topology, which is given by the $\mathbb{Z}_2$ discrete group. From the point of view of gravity, this symmetry provides a simple solution for the extrinsic curvature on the brane and enables the full construction of the Gauss–Codazzi formalism for a brane-world with one extra noncompact dimension [10]. This symmetry is also useful to analyze chiral fermions on the brane [11]. However, it is no longer a necessity in both problems. In the brane-world domain, there is a generalization of the Gauss–Codazzi formalism without a $\mathbb{Z}_2$ symmetry [12], and it seems that index theorems can lead to a good approach for the chirality problem. In other words, $\mathbb{Z}_2$ symmetry seems to be desirable but not indispensable.

On the other hand, Brans–Dicke gravity [13] is part (just by a relabeling of the Brans–Dicke parameter[1] w [14, 15]) of gravity recovered from string theory at low energy. In such a framework, recent advances using cosmic strings as topological defects to generate the compactification [17, 18] suggest that the resultant setup can be given in terms of a hybrid compactification scenario, i.e., when there is a noncompact extra dimension in the bulk and, at the same time, a compact one on the brane. So, in this picture we have compactification from six to five dimensions, where there is a small compact dimension S^1 at each point on the brane. This type of scenario is potentially interesting, because it can provide a good approach to explain the hierarchy problem, due to the hybrid compactification, and it also suggests a candidate for dark matter. However, this last claim needs further investigation in warped geometries.

As we mentioned above, the Brans–Dicke theory has a close relation with low energy gravitation recovered from string theory. Besides, the scalar field opens a new possibility in the scale adjustment of the Higgs mechanism. Apart from this, a rigorous formulation of brane-world models needs to treat the brane as a gravitational object with a nonzero tension. It is a requirement for the study of gravitational systems, as black holes for instance, on the brane. Of course, the presence of the new scalar field in the bulk can also bring new possibilities in such systems. In this vein, the manipulation and extension of the usual Gauss–Codazzi formalism to scalar tensorial theories is quite necessary. It was done to brane-worlds in usual general relativity [10], but the additional scalar field turns the manipulation of the equations more complicated and, as we will see later, introduces important differences in the final result.

This work is an attempt to establish the Gauss–Codazzi formalism for this kind of model. As we will see, the Brans–Dicke scalar field plays an important role in the effective cosmological constant on the brane, as well as in Newton's constant. In a few words, this work is a generalization of the Gauss–Codazzi equations for Brans–Dicke theory going from six to five dimensions inspired by the results found in [17, 18] and motivated by the above reasons. The paper is organized as follows: in Sect. 2 we apply the Gauss

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[1] In fact, such duality depends on the specific compactification scenario, see [16] for instance.

and the Codazzi equations for the brane-world in question and relate the Einstein equations on the brane with some bulk quantities using $\mathbb{Z}_2$ symmetry, in a similar fashion as [10]. In the last section we conclude by calling attention to the next step of this program, which is working without $\mathbb{Z}_2$ symmetry. In the appendix we show the matching condition for Brans–Dicke theory.

2 Gauss–Codazzi in the Brans–Dicke theory

We treat the brane as a submanifold, in five dimensions, out of a six dimensional manifold. It is important to remark that one of the five brane dimensions is compactified into a S^1 topology, so it is a codimension one scenario. The importance of this fact is that in higher codimension models the Gauss–Codazzi formalism is no longer useful,[2] since we need some minimal regularity near the brane in order to implement it [19, 20].

2.1 Notation and conventions

Basically we follow the notation and conventions used in [10]. The covariant derivative associated to the bulk is labeled by ∇_μ, while the one associated to the brane is D_μ. The induced metric on the brane is given by $q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, where n_μ is the unitary normal vector to the brane. In terms of this, the Gauss equation reads

$$^{(5)}R^{\alpha}_{\beta\gamma\delta} = {}^{(6)}R^{\mu}_{\nu\rho\sigma} q^{\alpha}_{\mu} q^{\nu}_{\beta} q^{\rho}_{\gamma} q^{\sigma}_{\delta} + K^{\alpha}_{\gamma} K_{\beta\delta} - K^{\alpha}_{\delta} K_{\beta\gamma} , \quad (1)$$

while the Codazzi equation is

$$D_\nu K^{\nu}_{\mu} - D_\mu K = {}^{(6)}R_{\rho\sigma} n^{\sigma} q^{\rho}_{\mu} , \quad (2)$$

where $K_{\mu\nu} = q^{\alpha}_{\mu} q^{\beta}_{\nu} \nabla_\alpha n_\beta$ is the extrinsic curvature. From (1) it is easy to find the Einstein equation in five dimensions in terms of the bulk quantities. The Ricci tensor is

$$^{(5)}R_{\beta\delta} = {}^{(6)}R_{\nu\sigma} q^{\nu}_{\beta} q^{\sigma}_{\delta} - {}^{(6)}R^{\mu}_{\nu\rho\sigma} n_\mu n^\rho q^{\nu}_{\beta} q^{\sigma}_{\delta} + K K_{\beta\delta} - K^{\gamma}_{\delta} K_{\beta\gamma}, \quad (3)$$

and the curvature scalar is given by

$$^{(5)}R = {}^{(6)}R_{\nu\sigma} q^{\nu\sigma} - {}^{(6)}R^{\mu}_{\nu\rho\sigma} n_\mu n^\rho q^{\nu\sigma} + K^2 - K^{\alpha\beta} K_{\alpha\beta} . \quad (4)$$

Then, the Einstein equation is given by

$$\begin{aligned} ^{(5)}G_{\beta\delta} = {}& {}^{(6)}G_{\nu\sigma} q^{\nu}_{\beta} q^{\sigma}_{\delta} + {}^{(6)}R_{\nu\sigma} n^{\nu} n^{\sigma} q_{\beta\delta} + K K_{\beta\delta} - K^{\gamma}_{\delta} K_{\beta\gamma} \\ & - \frac{1}{2} q_{\beta\delta} \left(K^2 - K^{\alpha\gamma} K_{\alpha\gamma}\right) - \tilde{E}_{\beta\delta} , \end{aligned} \quad (5)$$

where $\tilde{E}_{\beta\delta} = {}^{(6)}R^{\mu}_{\nu\rho\sigma} n_\mu n^\rho q^{\nu}_{\beta} q^{\sigma}_{\delta}$.

It is well known that the Riemann, Ricci and Weyl tensors are related among themselves, in an arbitrary number of dimensions (n), by

$$\begin{aligned} ^{(n)}R_{\alpha\beta\mu\nu} = {}& {}^{(n)}C_{\alpha\beta\mu\nu} + \frac{2}{n-2}\left({}^{(n)}R_{\alpha[\mu} g_{\nu]\beta} - {}^{(n)}R_{\beta[\mu} g_{\nu]\alpha}\right) \\ & - \frac{2}{(n-1)(n-2)} {}^{(n)}R g_{\alpha[\mu} g_{\nu]\beta} . \end{aligned} \quad (6)$$

[2] Apart from the regularization mechanism introduced in [12].

After some manipulation, the term $\tilde{E}_{\beta\delta}$ in (5) can be written as

$$\tilde{E}_{\beta\delta} = E_{\beta\delta} + \frac{1}{2}\left({}^{(6)}R^{\mu}_{\rho} n_\mu n^\rho q_{\beta\delta} + {}^{(6)}R_{\nu\sigma} q^{\nu}_{\beta} q^{\sigma}_{\delta}\right) - \frac{1}{10} {}^{(6)}R q_{\beta\delta}, \quad (7)$$

where $E_{\beta\delta} = {}^{(6)}C^{\mu}_{\nu\rho\sigma} n_\mu n^\rho q^{\nu}_{\beta} q^{\sigma}_{\rho}$. In terms of this new quantity (5) reads

$$\begin{aligned} ^{(5)}G_{\beta\delta} = {}& \frac{1}{2} {}^{(6)}G_{\nu\sigma} q^{\nu}_{\beta} q^{\sigma}_{\delta} - \frac{1}{10} {}^{(6)}R q_{\beta\delta} - \frac{1}{2} {}^{(6)}R_{\nu\sigma} q^{\nu\sigma} q_{\beta\delta} \\ & + K K_{\beta\delta} - K^{\gamma}_{\delta} K_{\beta\gamma} - \frac{1}{2} q_{\beta\delta} (K^2 - K^{\alpha\gamma} K_{\alpha\gamma}) - E_{\beta\delta} . \end{aligned} \quad (8)$$

2.2 Notation and conventions in Brans–Dicke theory

Now, the generalization to Brans–Dicke gravity means that one express the right-hand side of (8) in terms of the scalar field of such a theory. The Einstein–Brans–Dicke equation is

$$\begin{aligned} ^{(6)}G_{\mu\nu} = {}& \frac{8\pi}{\phi} T_{\mathrm{M}\mu\nu} + \frac{w}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha\phi\nabla^\alpha\phi\right) \\ & + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box^2\phi\right) , \end{aligned} \quad (9)$$

where ϕ is the Brans–Dicke scalar field, w a dimensionless parameter and $T_{\mathrm{M}\mu\nu}$ the matter energy-momentum tensor. Everything except ϕ and gravity, is in the bulk. The scalar equation of Brans–Dicke theory is given by

$$\Box^2\phi = \frac{8\pi}{3+2w} T_{\mathrm{M}} . \quad (10)$$

From (9) and (10) one can find the scalar of curvature of the bulk, as well as the Ricci tensor. The scalar has the form

$$^{(6)}R = \frac{-8\pi}{\phi}\left(\frac{w-1}{3+2w}\right) T_{\mathrm{M}} + \frac{w}{\phi^2} \nabla_\alpha\phi\nabla^\alpha\phi , \quad (11)$$

and the Ricci tensor

$$\begin{aligned} ^{(6)}R_{\mu\nu} = {}& \frac{8\pi}{\phi}\left[T_{\mathrm{M}\mu\nu} - \frac{1}{2} g_{\mu\nu}\left(\frac{1+w}{3+2w}\right) T_{\mathrm{M}}\right] \\ & + \frac{1}{\phi}\nabla_\mu\nabla_\nu\phi + \frac{w}{\phi^2}\nabla_\mu\phi\nabla_\nu\phi . \end{aligned} \quad (12)$$

Substituting (9), (11) and (12) into (8) we find the first step to the generalization of the Gauss–Codazzi formalism to the Brans–Dicke theory encoded in the following equation:

$$\begin{aligned} ^{(5)}G_{\beta\delta} = {}& \frac{1}{2}\left[\frac{8\pi}{\phi} T_{\mathrm{M}\nu\sigma} + \frac{1}{\phi}\nabla_\nu\nabla_\sigma\phi + \frac{w}{\phi^2}\nabla_\nu\phi\nabla_\sigma\phi\right] \\ & \times \left(q^{\nu}_{\beta} q^{\sigma}_{\delta} - q^{\nu\sigma} q_{\beta\delta}\right) \\ & + \frac{2\pi}{5\phi} q_{\beta\delta} T_{\mathrm{M}} \left(\frac{13+27w}{3+2w}\right) - \frac{7w}{20\phi^2} q_{\beta\delta} \nabla_\alpha\phi\nabla^\alpha\phi \\ & + K K_{\beta\delta} - K^{\gamma}_{\delta} K_{\beta\gamma} - \frac{1}{2} q_{\beta\delta}\left(K^2 - K^{\alpha\gamma} K_{\alpha\gamma}\right) - E_{\beta\delta}. \end{aligned} \quad (13)$$

The Codazzi equation (2) together with (12) gives

$$D_\nu K^\nu_\mu - D_\mu K = \left[\frac{8\pi}{\phi}T_{\mathrm{M}\rho\sigma} + \frac{1}{\phi}\nabla_\rho\nabla_\sigma\phi + \frac{w}{\phi^2}\nabla_\rho\phi\nabla_\sigma\phi\right] n^\sigma q^\rho_\mu\,. \tag{14}$$

Equations (13) and (14) summarize this stage. To extract information about the system, we have to compute some quantity on the brane, or rather, taking the limit of the extra dimensions tending to the brane. The extrinsic curvature is an important tool for the matching conditions. In order to guarantee sequential readability of the paper, we derive these equations in the appendix.

The effect of imposing the $\mathbb{Z}_2$ symmetry on the brane (or on each brane, in a multi-brane scenario) is to change the signal of the n_α vector across the brane, and consequently, to change the signal of the extrinsic curvature. Then, taking (A.14) into account we have

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = \frac{4\pi}{\phi}\left(-T_{\mu\nu} + \frac{q_{\mu\nu}(1+w)T}{2(3+2w)}\right) \tag{15}$$

and

$$K^+ = K^- = \frac{2\pi}{\phi}\left(\frac{w-1}{3+2w}\right)T\,. \tag{16}$$

Plugging (15) and (16) into (13) we have, after some algebra, the following result:

$$\begin{aligned}{}^{(5)}G_{\beta\delta} = {}& \frac{1}{2}\left[\frac{8\pi}{\phi}T_{\mathrm{M}\nu\sigma} + \frac{1}{\phi}\nabla_\nu\nabla_\sigma\phi + \frac{w}{\phi^2}\nabla_\nu\phi\nabla_\sigma\phi\right] \\ & \times\left(q^\nu_\beta q^\sigma_\delta - q^{\nu\sigma}q_{\beta\delta}\right) \\ & + \frac{2\pi}{5\phi}q_{\beta\delta}T_\mathrm{M}\left(\frac{13+27w}{3+2w}\right) - \frac{7w}{20\phi^2}q_{\beta\delta}\nabla_\alpha\phi\nabla^\alpha\phi \\ & + 8\left(\frac{\pi}{\phi}\right)^2\left[TT_{\beta\delta}\left(\frac{w+3}{3+2w}\right) - T^2 q_{\beta\delta}\frac{(w^2+3w+3)}{(3+2w)^2}\right. \\ & \left. - 2T^\gamma_\delta T_{\beta\gamma} + q_{\beta\delta}T^{\alpha\gamma}T_{\alpha\gamma}\right] - E_{\beta\delta}\,. \end{aligned} \tag{17}$$

Note that the values of $E_{\mu\nu}$ and the Brans–Dicke field, as well as their derivatives, are not taken exactly on the brane, but in the limit $y \to 0^\pm$. Now we split the matter energy-momentum in the form [10]

$$T_{\mathrm{M}\mu\nu} = -\Lambda g_{\mu\nu} + \delta(y)T_{\mu\nu}\,, \tag{18}$$

and

$$T_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu}\,, \tag{19}$$

where Λ is the cosmological constant of the bulk and λ the tension of the brane. It should be stressed that such a type of decomposition may lead to some ambiguity in the cosmological scenario. However, here it can be done and, actually, it is quite useful to interpret the final result. So, putting (18) and (19) inside (17) we obtain

$$\begin{aligned}{}^{(5)}G_{\beta\delta} = {}& \frac{1}{2}\left[\frac{1}{\phi}\nabla_\nu\nabla_\sigma\phi + \frac{w}{\phi^2}\nabla_\nu\phi\nabla_\sigma\phi\right]\left(q^\nu_\beta q^\sigma_\delta - q^{\nu\sigma}q_{\beta\delta}\right) \\ & + 8\pi\Omega\tau_{\beta\delta} - \Lambda_5 q_{\beta\delta} + 8\left(\frac{\pi}{\phi}\right)^2\Sigma_{\beta\delta} - E_{\beta\delta}\,, \end{aligned} \tag{20}$$

where

$$\Omega = \frac{3\pi(w-1)\lambda}{\phi^2(3+2w)}\,, \tag{21}$$

$$\begin{aligned}\Lambda_5 = {}& \frac{-4\pi\Lambda(21-41w)}{5\phi(3+2w)} \\ & + \left(\frac{\pi}{\phi}\right)^2\left[\frac{7w}{20\pi^2}\nabla_\alpha\phi\nabla^\alpha\phi + \frac{24(w-1)\lambda}{(3+2w)^2}[(w-1)\lambda+\tau]\right]\end{aligned} \tag{22}$$

and

$$\begin{aligned}\Sigma_{\beta\delta} = {}& q_{\beta\delta}\tau^{\alpha\gamma}\tau_{\alpha\gamma} - 2\tau^\gamma_\delta\tau_{\gamma\beta} + \left(\frac{3+w}{3+2w}\right)\tau\tau_{\beta\delta} \\ & - \frac{(w^2+3w+3)}{(3+2w)^2}q_{\beta\delta}\tau^2\,. \end{aligned} \tag{23}$$

Let us analyze (20) in more detail. First of all, we do not write it in a Brans–Dicke form by the simple fact that Brans–Dicke theory is not recovered on the brane in models in which the scalar field depends only on the extra dimension [17, 18]. We shall restrict the analysis to such cases. So it seems more plausible to look for deviations of the usual Einstein brane-worlds formulation; something like "effective" Einstein equations on the brane. The first term arises from the scalar field contribution and it brings us information about the bulk structure. Note that from the point of view of a brane observer, such a field has no dynamics, since the brane is localized at a fixed y. In the second term, there is a type of effective Newton's coupling constant, which depends on the scalar field, as well as on the brane tension. Since $\Omega = \Omega(\phi(r))$ the scalar field must be stabilized in order to guarantee the agreement with usual gravity on the brane. Artificially it can be understood as an adjustment of the brane along to the extra dimension, inducing the right value to the scalar field. Strictly speaking, it should be done by the introduction of a well-behaved potential in the Brans–Dicke part of the action, see for instance [16].

As in [10], here we recover the fact that it is not possible to define a gravitational constant in an era where there was no distinction between vacuum energy and the usual matter energy. Besides, the signal of Ω strongly depends on the signal of the brane tension. The equation (22) calls our attention to the effective cosmological constant in five dimensions. It depends on the bulk cosmological constant and on the scalar field; hence it stresses the fact that the cosmological constant may be variable for a bulk observer. The penultimate factor is quadratic in the energy-momentum on the brane and could, hence, be an important part of cosmological evolution in the early universe (see, for example, [21] for the five dimensional case). The last term provides more information on the gravitational field of the bulk, which justifies the inclusion of Weyl's tensor in the analysis.

Returning to the first term of (20), inserting (15) and (16) into (14) and taking into account the split defined by (18) and (19), the Codazzi equation gives an important relation between the scalar field and derivatives of $\tau_{\mu\nu}$:

$$\frac{4\pi}{\phi}\left[-D_\nu \tau^\nu_\mu + \frac{1}{3+2w} D_\mu \tau\right] = \left[\frac{1}{\phi}\nabla_\rho\nabla_\sigma\phi + \frac{w}{\phi^2}\nabla_\rho\phi\nabla_\sigma\phi\right] n^\sigma q^\rho_\mu\,. \quad (24)$$

As a last remark, we emphasize that the term $E_{\mu\nu}$, which encodes the Weyl bulk tensor, also has a restricted divergence by derivatives of $\tau_{\mu\nu}$, as we can see by the contracted Bianchi identities $D^{\beta(5)}G_{\beta\delta} = 0$ applied to (20),

$$D^\beta E_{\beta\delta} = \frac{8\pi}{\phi} D^\beta \tau_{\beta\delta} - \frac{24\pi(w-1)\lambda}{(3+2w)^2} D_\delta \tau + 8\left(\frac{\pi}{\phi}\right)^2 D^\beta \Sigma_{\beta\delta}\,. \quad (25)$$

Obviously the two last equations can be used to relate the Brans–Dicke field with the Weyl tensor. However, it could not be desirable if one wants to work with the extensive technology developed for the $E_{\mu\nu}$ tensor (see Appendix A of [10]).

3 Conclusion

In this work we generalized the Gauss–Codazzi formalism for brane-worlds in a Brans–Dicke gravity framework. It was done in the context of $\mathbb{Z}_2$ symmetry models, which allows us to uniquely determine the extrinsic curvature on the brane. The main result obtained is (20), which resembles the results obtained for Einstein's theory, but it brings about important differences. The Brans–Dicke scalar field is present in all terms of the right-hand side except in $E_{\mu\nu}$. Keeping in mind the first term of (20) together with (24) and (25), it seems to be almost certain that we will find shunting lines in the study of gravitational physical systems on the brane like, for example, the black hole area and the quasar luminosity [24]. We shall address those questions in the future.

Two interesting points call our attention in (20). First, we notice the coupling between the brane tension λ and the Brans–Dicke parameter w. In the standard formulation of the Brans–Dicke theory such a parameter can be expected to be of order of unity.[3] By (21) and (22) it is clear that in the case of $w = 1$ the effective Newton constant vanishes, which is a negative result, and there is no contribution of the brane tension in (20). This type of inconsistence persists in the case without $\mathbb{Z}_2$ symmetry [23]. However, we should stress that it is a possible inconsistence between pure Brans–Dicke theory and brane-world models; but this is not the case here, since we are using such a theory in order to mimic low energy gravity recovered from string theory. The second point is that there is a configuration of the scalar field in which the induced cosmological constant on the brane vanishes. From (22), assuming that the bulk cosmological constant is constant, it is easy to see that $\Lambda_5 = 0$ if the scalar field has the form

$$\phi(y) = \frac{4BC + \Lambda^2 A^2 (y-D)^2}{4AB\Lambda}\,, \quad (26)$$

where D is a constant of integration and $A = \frac{4\pi(21-41w)}{5(3+2w)}$, $B = \frac{7w}{20}$ and $C = \frac{24\pi^2(w-1)\lambda[(w-1)\lambda+\tau]}{(3+2w)^2}$. We note that Λ_5 also vanishes for $\phi(y) = \frac{C}{A\Lambda}$. For nonconstant Λ, it is necessary to have the explicit behavior of the function in order to do a similar analysis. Note that the constant solution for the scalar field is not of physical interest for this type of extension. The polynomial solution (26) is not usual in the sense of models like [17, 18]. However, it is an important result, since it can say, in the scope of the models in question, what type of behavior of the Brans–Dicke scalar field can lead to an effective cosmological constant on the brane.

It is important to notice the results obtained in the appendix. It is a direct generalization of the matching conditions to the Brans–Dicke case. To conclude, we discuss a little bit more the role played by the $\mathbb{Z}_2$ symmetry. The model we consider has one compact on brane dimension. It is also interesting to analyze this type of models without the $\mathbb{Z}_2$ symmetry [12], since without such a simplification the final equations show a new term, which arises from the mean of extrinsic curvature; it leads to anisotropic matter on the brane and then could be useful to interpret the hybrid brane-world compactification scenario.

Acknowledgements. The authors benefited from useful discussions with Profs. Andrey Bytsenko, José Helayël-Neto and Hiroshi de Sandes Kimura. The authors are also grateful to the EJPC referee for useful comments and enlightening viewpoints. J.M. Hoff da Silva thanks CAPES-Brazil for financial support. M.E.X. Guimarães and M.C.B. Abdalla acknowledge CNPq for support.

Appendix: Israel–Darmois matching conditions to Brans–Dicke gravity

In order to extend the usual junction conditions [25] to the case in question we use distributional calculus, just like in [26]. The basic approach is to treat the brane as a (infinitely thin) hypersurface orthogonally riddled by geodesics. Denoting the extra dimensional coordinate by y, it is always possible to choose some parametrization in which the brane is located at $y = 0$, in such a way that $y > 0$ represents one side of the brane and $y < 0$ the other side. In this appendix we use the notation

$$[\chi] = \chi^+ - \chi^-\,, \quad (A.1)$$

[3] Meanwhile experiments show that this is not the case. See for instance [22] for the current lower bound of the Brans–Dicke parameter.

where the $\chi^{\pm}$ denote the limits of the χ quantity approaching the brane when $y \to 0^{\pm}$. So, it is possible to analyze the jump of χ through the hypersurface. With the Heaviside distribution $\Theta(y)$ it is possible to decompose quantities at both sides of the brane. The bulk metric can then be expressed as

$$g_{\mu\nu} = \Theta(y) g^{+}_{\mu\nu} + \Theta(-y) g^{-}_{\mu\nu} \,, \tag{A.2}$$

where $g^{+}_{\mu\nu}$ ($g^{-}_{\mu\nu}$) is the metric on the right- (left-) hand side. Note that the Heaviside distribution has the following properties: $\Theta(y) = +1$ if $y > 0$, indeterminate if $y = 0$ and zero otherwise. Besides

$$\Theta^2(y) = \Theta(y) \,, \quad \Theta(y)\Theta(-y) = 0 \,, \quad \frac{\mathrm{d}\Theta(y)}{\mathrm{d}y} = \delta(y) \,, \tag{A.3}$$

where $\delta(y)$ is the Dirac distribution. With such tools it is easy to note that

$$g_{\mu\nu,\alpha} = \Theta(y) g^{+}_{\mu\nu,\alpha} + \Theta(-y) g^{-}_{\mu\nu,\alpha} + \delta(y)[g_{\mu\nu}] n_\alpha \,, \tag{A.4}$$

and since the Christoffel symbols constructed with (A.4) will generate products as $\Theta(y)\delta(y)$, which is not well defined as a distribution, one is forced to conclude that $[g_{\mu\nu}] = 0$. This is the so-called Darmois condition. In this appendix we use the compact notation $A_{,\mu} = \nabla_\mu A$, A being any tensorial quantity.

Following this reasoning and supposing that the discontinuity of $g_{\mu\nu,\alpha}$ is directed along n^α by $[g_{\mu\nu,\alpha}] = k_{\mu\nu} n_\alpha$, for some tensor $k_{\mu\nu}$, one finds that the left-hand side (the geometrical part) of the Einstein's tensor can be decomposed in a such way that its δ-function part (the part on the brane itself) is given by [26]

$$\frac{1}{2}\big(k_{\gamma\mu} n^\gamma n_\nu + k_{\gamma\nu} n^\gamma n_\mu - k n_\mu n_\nu - k_{\mu\nu} - \left(k_{\gamma\sigma} n^\gamma n^\sigma - k\right) g_{\mu\nu}\big) \equiv S_{\mu\nu} \,. \tag{A.5}$$

Let us now look at the Brans–Dicke part of the decomposition. Writing the scalar field as

$$\phi = \Theta(y)\phi^{+} + \Theta(-y)\phi^{-} \,, \tag{A.6}$$

we have

$$\phi_{,\mu} = \Theta(y)\phi^{+}_{,\mu} + \Theta(-y)\phi^{-}_{,\mu} + \delta(y)[\phi] n_\mu \,. \tag{A.7}$$

Since the Brans–Dicke equation is given by (9) one has to impose continuity of the field across the brane, i.e., $[\phi] = 0$, in order to avoid $\Theta(y)\delta(y)$ terms arising from the second term of the right-hand side of (9), just as we did before. This is the analogous to the Darmois junction condition in the Brans–Dicke case.

To find another matching condition on the brane, which involves the extrinsic curvature, we need to decompose all terms of the right-hand side of (9) and equalize it to $S_{\mu\nu}$ (A.5). From (A.7) we have

$$\phi_{,\mu;\nu} = \Theta(y)\phi^{+}_{,\mu;\nu} + \Theta(-y)\phi^{-}_{,\mu;\nu} + \delta(y)[\phi_{,\mu}] n_\nu \,, \tag{A.8}$$

while the energy-momentum tensor is decomposed as

$$T^{\text{total}}_{\mu\nu} = \Theta(y) T^{+}_{\mu\nu} + \Theta(-y) T^{-}_{\mu\nu} + \delta(y) T_{\mu\nu} \,, \tag{A.9}$$

where $T_{\mu\nu}$ is the energy-momentum on the brane. Before substituting these factors into (9) we have to deal with the scalar fields in the denominator. Note that in the standard distributional decomposition it is not difficult to see that

$$\frac{1}{\phi} = \frac{\Theta(y)}{\phi^{+}} + \frac{\Theta(-y)}{\phi^{-}} \,. \tag{A.10}$$

The $1/\phi^2$ factor obeys a similar splitting. So, after substituting all decompositions into (9) one finds that the δ-function part of the Brans–Dicke term is

$$\frac{8\pi}{\phi}\left(T_{\mu\nu} - \frac{g_{\mu\nu}}{3+2w} T\right) + \frac{1}{\phi}[\phi_{,\mu}] n_\nu \equiv (\mathrm{BD})_{\mu\nu} \,. \tag{A.11}$$

Obviously $S_{\mu\nu} = (\mathrm{BD})_{\mu\nu}$, and since $S_{\mu\nu} n^\nu = 0$ (from (A.5)) we arrive at

$$[\phi_{,\mu}] = \frac{8\pi}{3+2w} T n_\mu \,, \tag{A.12}$$

since the contraction $T_{\mu\nu} n^\nu$ is zero (because $T_{\mu\nu}$ belongs to the brane). Equation (A.12) is the analog of $[g_{\mu\nu,\alpha}] = k_{\mu\nu} n_\alpha$. Now substituting this result into $(\mathrm{BD})_{\mu\nu}$ we determine completely the δ-function term of the Brans–Dicke equations, which is

$$(\mathrm{BD})_{\mu\nu} = \frac{8\pi}{\phi}\left(T_{\mu\nu} - q_{\mu\nu}\frac{T}{3+2w}\right) . \tag{A.13}$$

The $S_{\mu\nu}$ term of Einstein's equation can be related with the jump of the extrinsic curvature across the hypersurface [26] by $-([K_{\mu\nu}] - [K] q_{\mu\nu})$; then, after all this, we have the analog to the second matching condition for the Brans–Dicke case given by

$$\frac{8\pi}{\phi}\left(T_{\mu\nu} - q_{\mu\nu}\frac{T}{3+2w}\right) = -([K_{\mu\nu}] - [K] q_{\mu\nu}) \,. \tag{A.14}$$

This expression was obtained within the brane-world context. However, it is still valid for the usual four dimensions, if the scalar field depends only on the transverse direction to the hypersurface. We stress that, as expected, if $\phi = 1/G$ and $w \to \infty$ the expression for the Israel matching condition in general relativity is recovered.

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